Bifurcations of the geometric phase of a qubit asymmetrically coupled to the environment

2008 J. Phys. A: Math. Theor. 41 442001

(http://iopscience.iop.org/1751-8121/41/44/442001)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **41** (2008) 442001 (8pp) doi:10.1088/1751-8113/41/44/442001

FAST TRACK COMMUNICATION

Bifurcations of the geometric phase of a qubit asymmetrically coupled to the environment

Jerzy Dajka and Jerzy Łuczka

Institute of Physics, University of Silesia, 40-007 Katowice, Poland

Received 21 August 2008
Published 7 October 2008
Online at stacks.iop.org/JPhysA/41/442001

Abstract
The geometric phase of a qubit asymmetrically coupled to the outer bosonic environment is studied. We demonstrate that with the change of the coupling asymmetry, the geometric phase can exhibit a cascade of bifurcations and therefore can be useful for testing the asymmetry coupling.

PACS numbers: 03.65.Yz, 03.65.Vf, 03.67.−a

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Quantum computation and effective quantum information storage require methods and tools to overcome, or at least to reduce, decoherence processes in quantum information units which are always influenced by the environment. In recent years, this problem has been intensively investigated in a variety of contexts and aspects. The coupling of quantum units (qubit, qutrits, etc) to the outer environment can be symmetric or asymmetric. The question is whether the coupling symmetry breaking can be recognized or maybe it does not matter what symmetry of coupling is. We pose this problem and want to answer the above question. To this aim, we consider a qubit (a two-level system) coupled to a bosonic environment. We analyze a particular example of environmental effects: dephasing or pure decoherence. The coupling is via an integral of motion and in consequence the system does not exchange energy with the environment. The only exchange with the environment is information. We extend our previous study [1] to the case when each level of a qubit can be coupled to the environment in a different fashion, i.e. non-symmetrically. It allows us to consider a class of qubit–environment Hamiltonians ranging from the most popular one, the isotropic van Hove model [2], up to the Friedrichs model [3] for which only one of the two levels is coupled to the environment. We study properties of the geometric phase (GP), gained by the qubit in a (quasi) cyclic evolution. In some regimes, the GP is sensitive to the symmetry of the coupling: for symmetric coupling, the GP is a monotonic function of the angle (which defines the initial state of the qubit as a linear combination of two canonical basic states of the qubit); for asymmetric coupling, it is a non-monotonic function of the angle and can exhibit bifurcations when the asymmetry coupling is changed. Therefore in such regimes, the GP can be a tool for detecting and quantifying the asymmetry of the dephasing environment, i.e. the GP is an 'observable' for testing details of microscopic modeling at the Hamiltonian level. In other regimes, the GP does not change its monotonicity with respect to the angle.

Studies of a GP of a quantum system start from the classical Pancharatnam notion and quantum Berry phase [4] factor in the adiabatic and cyclic unitary evolution of non-degenerate states. There are plenty of generalizations including nonadiabatic [5], noncyclic and even nonunitary evolution of the quantum state. As a general reference one can consult, e.g. [6]. The extension to the mixed states has been proposed first in [7] in a purely mathematical fashion and in [8] for unitary evolutions with a clear interferometric interpretation. At the same time the notion of GP becomes more and more attractive for quantum information due to its possible application for geometric or holonomic quantum computation as a means of constructing built-in fault tolerant quantum logic gates [9–11]. As it is much less dependent on the details of the time evolution it may be less affected by uncontrolled fluctuations and therefore more robust against certain sources of perturbations.

2. Asymmetric dephasing

We consider a qubit Q, formed by an arbitrary two-level system (e.g. a spin-$1/2$ particle) for which $\{|1\rangle, |-1\rangle\}$ is its canonical basis. The qubit is coupled to the environment R. The coupling is such that there is no energy dissipation. However, there is an irreversible process of information loss [19]. It can be modeled by the Hamiltonian [12, 13]

$$H = H_Q + H_R + H_I, \qquad H_Q = \epsilon S^z \otimes \mathcal{I}_R, \qquad H_R = \mathcal{I}_Q \otimes H_B, \tag{1}$$

where the qubit Hamiltonian H_Q is defined by the z-component of the spin operator $S^z = |1\rangle\langle 1| - |-1\rangle\langle -1|$ and the energy levels are $E = \pm\epsilon$. The operators $\mathcal{I}_Q$ and $\mathcal{I}_R$ are identity operators (matrices) in the corresponding Hilbert spaces of the qubit and the environment, respectively. The environment is composed of bosons for which the Hamiltonian H_B reads

$$H_B = \int_0^\infty \mathrm{d}\omega\, h(\omega) a^\dagger(\omega) a(\omega), \tag{2}$$

where the real-valued dispersion relation $h(\omega)$ specifies the environment, $a^\dagger(\omega)$ and $a(\omega)$ are the creation and annihilation boson operators, respectively. The qubit–environment interaction is described by the Hamiltonian

$$H_I = |1\rangle\langle 1| \otimes H_+ + |-1\rangle\langle -1| \otimes H_-, \tag{3}$$

$$H_\pm = \pm \int_0^\infty \mathrm{d}\omega [g_\pm^*(\omega) a(\omega) + g_\pm(\omega) a^\dagger(\omega)], \tag{4}$$

where $g_\pm(\omega)$ are the coupling functions and $g_\pm^*(\omega)$ are the complex conjugate functions to $g_\pm(\omega)$, respectively. The Hamiltonian (1)–(4) can be rewritten in the form

$$H = |1\rangle\langle 1| \otimes H_1 + |-1\rangle\langle -1| \otimes H_{-1}, \qquad H_{1/-1} = H_B + H_\pm \pm \varepsilon. \tag{5}$$

Hamiltonians like (5) has been used to study the electron-transfer reactions [14] and the interconversion of electronic and vibrational energy [15]. With a similar structure, they have been considered to analyze a quantum kicked rotator [16], chaotic dynamics of a periodically driven superconducting single electron transistor [17] and the Josephson flux qubit [18]. The model may also serve as a component of a simple quantum register [19]. Moreover, it contains, as particular cases, the widely used van Hove model [2, 13, 20] (for $g_+ = g_-$) and the Friedrichs model [3] (for either $g_+ = 0$ or $g_- = 0$).

We can directly apply results of [21] to the Hamiltonian (5) and solve the corresponding Schrödinger equation. To do it, let us specify an initial state of the system assuming a product state, namely,

$$|\Psi(0)\rangle = (b_1|1\rangle + b_{-1}|-1\rangle) \otimes |R\rangle, \tag{6}$$

where b_1 and b_{-1} determine the qubit states and $|R\rangle$ is the initial state of the environment with the Hamiltonian (1). The state (6) evolves as follows [21]:

$$\begin{aligned}|\Psi(t)\rangle = b_1\, \mathrm{e}^{-\mathrm{i}\Lambda_1(t)}|1\rangle \otimes D\big(g_1^t - g_1\big)\, \mathrm{e}^{-\mathrm{i}H_B t}|R\rangle,\\ + b_{-1}\, \mathrm{e}^{-\mathrm{i}\Lambda_{-1}(t)}|-1\rangle \otimes D\big(g_{-1} - g_{-1}^t\big)\, \mathrm{e}^{-\mathrm{i}H_B t}|R\rangle,\end{aligned} \tag{7}$$

where the phases $\Lambda_1(t)$ and $\Lambda_{-1}(t)$ have the form

$$\begin{aligned}\Lambda_1(t) &= \varepsilon t - \int_0^\infty \mathrm{d}\omega |g_1(\omega)|^2\{h(\omega)t - \sin[h(\omega)t]\},\\ \Lambda_{-1}(t) &= -\varepsilon t - \int_0^\infty \mathrm{d}\omega |g_{-1}(\omega)|^2\{h(\omega)t - \sin[h(\omega)t]\}\end{aligned} \tag{8}$$

and the abbreviations

$$g_1(\omega) = \frac{g_+(\omega)}{h(\omega)}, \qquad g_{-1}(\omega) = \frac{g_-(\omega)}{h(\omega)} \tag{9}$$

have been introduced. For any function g, the notation g^t means $g^t(\omega) = \mathrm{e}^{-\mathrm{i}h(\omega)t}g(\omega)$. The displacement operator $D(g) = \exp[a^\dagger(g) - a(g)]$ [20, 22, 23], where $a(g) = \int_0^\infty \mathrm{d}\omega\, g(\omega)a(\omega)$ for an arbitrary square-integrable function g.

3. Exact reduced dynamics

We are not interested in full information on the total system: qubit + environment. Rather influence of the environment and dynamics of the qubit alone is desired. The qubit dynamics can be obtained for initial states like (6) or for a larger class of states defined by the initial statistical operator (density matrix) $\varrho(0)$ of the total system,

$$\varrho(0) = \sum_{i,j=1,-1} p_{ij}|\phi_i\rangle\langle\phi_j| \otimes |R\rangle\langle R|, \tag{10}$$

where $|\phi_i\rangle = |i\rangle, i = \pm 1$ are vectors of the qubit Hilbert space and p_{ij} are non-negative parameters. The reduced statistical operator $\rho(t)$ for the qubit only can be expressed in the forms

$$\begin{aligned}\rho(t) = \mathrm{Tr}_R[\varrho(t)] &= \sum_{i,j=1,-1} p_{ij}|\phi_i\rangle\langle\phi_j| \otimes \mathrm{Tr}_R(\mathrm{e}^{-\mathrm{i}H_i t}|R\rangle\langle R|\,\mathrm{e}^{\mathrm{i}H_j t})\\ &= \sum_{i,j=1,-1} p_{ij}\langle\, \mathrm{e}^{-\mathrm{i}H_j t}R|\,\mathrm{e}^{-\mathrm{i}H_i t}R\rangle|\phi_i\rangle\langle\phi_j|,\end{aligned} \tag{11}$$

where Tr_R denotes partial tracing over the environment. So, if we are able to calculate the scalar product in the Hilbert space of environment then the reduced dynamics is exactly constructed. We consider the simplest case assuming the initial state to be a vacuum, i.e. $|R\rangle = |\Omega\rangle$. For the initial qubit state

$$|\phi(0)\rangle = \cos(\theta/2)|1\rangle + \sin(\theta/2)|-1\rangle \tag{12}$$

parametrized by the angle θ, the statistical operator $\rho(t)$ for the reduced qubit dynamics takes the matrix form

$$\rho(t) = \begin{pmatrix} \cos^2(\theta/2) & (1/2)A(t)\sin\theta \\ (1/2)A^*(t)\sin\theta & \sin^2(\theta/2) \end{pmatrix}. \tag{13}$$

This formula defines the *asymmetric* dephasing channel $\mathcal{E} : \rho(0) \to \rho(t)$, where the influence of the infinite bosonic environment is represented by the function

$$A(t) = A_d(t)\exp[-\mathrm{i}\Xi(t)]. \tag{14}$$

The phase part

$$\Xi(t) = \Lambda_1(t) - \Lambda_{-1}(t). \tag{15}$$

The damping part reads

$$A_d(t) = \langle\Omega|D\big(g_1^t - g_1 + g_{-1}^t - g_{-1}\big)|\Omega\rangle = \exp[-r(t)] \tag{16}$$

with

$$r(t) = \int_0^\infty \mathrm{d}\omega|g_1(\omega) + g_{-1}(\omega)|^2\{1 - \cos[h(\omega)t]\}. \tag{17}$$

In the following we assume the 'linear' environment defined by the dispersion relation $h(\omega) = \omega$. The coupling to the environment is encoded in the coupling functions $g_i(\omega), i = 1, -1$. For convenience, we can assume that they are real functions. The form of the damping function (17) suggests that the interaction can be modeled by the spectral densities [12]

$$J_i(\omega) = \omega^2 g_i(\omega)^2 = \lambda_i\omega^{1+\mu_i}\exp\left(-\omega/\omega_i^c\right) \tag{18}$$

for $i = 1, -1$ and $\mu_i > -1$. The above form of the spectral densities $J_i(\omega)$ is well known in the literature. The case $\mu_i \in (-1, 0)$ corresponds to sub-Ohmic, $\mu_i = 0$ to Ohmic and $\mu_i \in (0, \infty)$ to super-Ohmic environments. To avoid mathematical difficulties mentioned in [13] we limit our discussion to the super-Ohmic environment where there is no controversy concerning the existence of the ground state. Then the damping function takes the form

$$r(t) = \mathcal{L}(\lambda_1, \mu_1; t) + \mathcal{L}(\lambda_{-1}, \mu_{-1}; t) + 2\mathcal{L}\left(\sqrt{\lambda_1\lambda_{-1}}, \frac{\mu_1 + \mu_{-1}}{2}; t\right), \tag{19}$$

where

$$\mathcal{L}(\lambda, \mu; t) = \lambda\Gamma(\mu)\omega_c^\mu\left[1 + \frac{\cos(\mu\arctan(\omega_c t))}{\left(1 + \omega_c^2 t^2\right)^{\mu/2}}\right]$$

and $\Gamma(z)$ is the Euler gamma function. The oscillatory part introduced by the environment can also be explicitly evaluated reading

$$\Xi(t) = 2\varepsilon t + \mathcal{M}(\lambda_1, \mu_1; t) + \mathcal{N}(\lambda_1, \mu_1; t) - \mathcal{M}(\lambda_{-1}, \mu_{-1}; t) - \mathcal{N}(\lambda_{-1}, \mu_{-1}; t), \tag{20}$$

where

$$\mathcal{M}(\lambda, \mu; t) = \lambda\omega_c^{1+\mu}\Gamma(1 + \mu)t, \qquad \mathcal{N}(\lambda, \mu; t) = \frac{\lambda\Gamma(\mu)\omega_c^\mu\sin(\mu\arctan(\omega_c t))}{\left(1 + \omega_c^2 t^2\right)^{\mu/2}}. \tag{21}$$

It finishes a presentation of all elements of the exact reduced dynamics for the qubit.

4. Geometric phase

There are several extensions of the GP for open systems. They are based on the state purification, quantum trajectories and quantum interferometry (kinematic approach). Here, we follow the approach based on the state purification [24]. The reduced density matrix $\rho(t)$ of the qubit given by equation (13) can be presented in the form

$$\rho(t) = \sum_{i=1}^{2} p_i(t)|w_i(t)\rangle\langle w_i(t)|, \tag{22}$$

where $p_i(t)$ and $|w_i(t)\rangle$ are eigenvalues and eigenvectors of the matrix (13). The GP $\Phi(t)$ associated with such an evolution is defined in the following way [24]:

$$\Phi(t) = \arg\left[\sum_{i=1}^{2}[p_i(0)p_i(t)]^{1/2}\langle w_i(0)|w_i(t)\rangle \exp\left(-\int_0^t \langle w_i(s)|\dot{w}_i(s)\rangle\,\mathrm{d}s\right)\right]. \tag{23}$$

This phase can be measured in some experiments [24].

The evolution of the freely evolving qubit is cyclic with time $T = \pi/\varepsilon$. In the presence of the dephasing environment the evolution becomes non-unitary and quasi-periodic. In this section we discuss the geometric phase $\Phi(t)$ of the qubit initially prepared in the state (12) calculated at time $t = T$ for $\varepsilon = 1/2$, i.e. $\Phi = \Phi(2\pi)$. We focus on the role of the coupling asymmetry. This asymmetry can be parametrized in various ways. Here, it is described by the parameter δ in the following way:

$$\lambda_{\pm 1} = \lambda[1 \pm \delta/100]. \tag{24}$$

For $\delta = 0$, the coupling is symmetric giving the van Hove model while for $\delta = 100$ the model reduces to the fully asymmetric, Friedrichs model. First, we consider the symmetric coupling $\lambda_{\pm 1} = \lambda$ and investigate the role of the coupling strength λ, see the left panel of figure 1. For the non-coupled case ($\lambda = 0$), the GP reads [6]

$$\Phi_0 = \pi[\cos(\theta) + 1]. \tag{25}$$

For weak coupling, the GP is close to that for the non-coupled (isolated) qubit, see the cases $\lambda = 0.001$ and $\lambda = 0.01$ in the left panel of figure 1. For strong coupling (the case $\lambda = 0.1$ in the left panel of figure 1), the GP changes drastically in the region $\theta = \pi/2$ and changes weakly outside this region. Similar behavior is observed when the low-frequency properties of the environment are changed. They are encoded in the parameters $\mu_{\pm 1} = \mu$. For small values of μ (weakly super-Ohmic environment or closed to the Ohmic one), the GP is close to that for the non-coupled qubit, see the cases $\mu = 0.001$ and $\mu = 0.1$ in the right panel of figure 1. For strongly super-Ohmic environment, the GP changes drastically in the region $\theta = \pi/2$ and changes weakly outside this region, see the case $\mu = 1$ in the right panel of figure 1. The case of asymmetrical coupling is much more interesting: the GP can be a non-monotonic function and even can jump in dependence of the initial state of the qubit, i.e. of the angle θ, cf the left panel of figure 2. Moreover, the GP as a function of the asymmetry parameter δ can exhibit bifurcations. It is presented in the right panel of figure 2 where the pitchfork bifurcation can be noted. At the bifurcation point $\delta = \delta_1$, the number of jumps of the GP changes from one to two. The critical value δ_1 is smaller for stronger coupling λ. If the coupling constant λ is increased, the second pitchfork bifurcation occurs and the second bifurcation point $\delta = \delta_2 > \delta_1$ can be observed, see the left panel of figure 3. For $\delta > \delta_2$, there are three jumps of the GP. Further increase the coupling strength λ results in a cascade of bifurcations, see the right panel of figure 3. It would be interesting to perform a detailed analysis (if possible at all) of bifurcations and answer several questions like: it is a finite

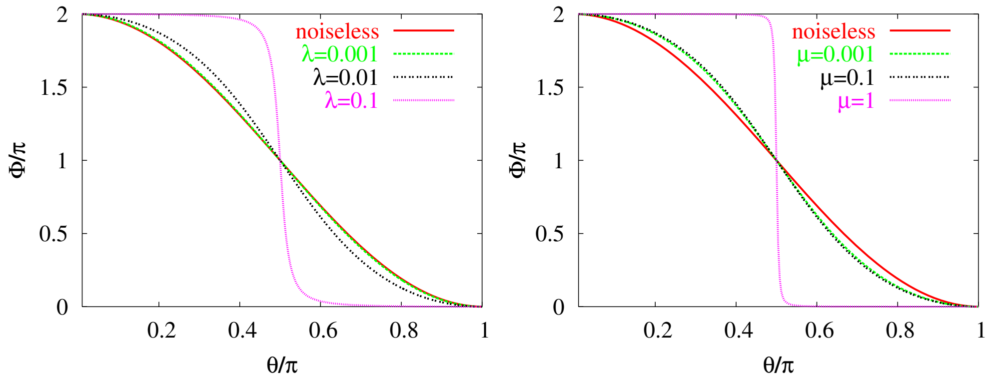

Figure 1. The geometric phase $\Phi = \Phi(2\pi)$ of the qubit in the presence of symmetric coupling to the environment (the van Hove model) compared with the geometric phase of the freely evolving qubit ($\lambda = 0$). Left panel: fixed $\mu_{\pm} = 0.1$ and selected values of the coupling strength λ. Right panel: fixed $\lambda_{\pm 1} = 0.01$ and selected values of $\mu_{\pm} = \mu$ which describes low-frequency properties of the environment: the super-Ohmic case.

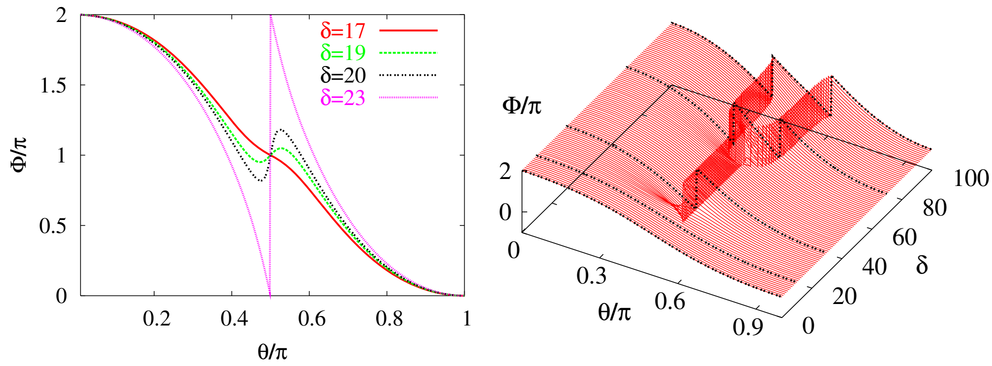

Figure 2. The geometric phase $\Phi = \Phi(2\pi)$ for the asymmetric coupling of the qubit to the environment. The asymmetry is parametrized by δ according to equation (24). Left panel: details of formation of the first jump. Right panel: the first bifurcation and formation of jumps with increasing asymmetry δ. The selected curves correspond to $\delta = 0, 10, 30, 70, 100$. $\lambda = 0.008$, $\mu_{\pm 1} = 0.1$.

or infinite number of jumps (bifurcation points) as λ grows or can the GP exhibit chaotic properties. Although it has nothing in common with the Feigenbaum scenario of chaos, maybe some similarities are hidden in the GP properties. We leave it as an open problem. The next open problem is related to possible universal properties of jumps. As they form a cascade of bifurcations, to what extent they appear regularly with respect to e.g. different, comparing to those chosen in equation (24), parametrization of the asymmetry of coupling.

5. Summary

The GP is an experimentally observable quantity (although not an observable in a strict sense) which can detect possible anisotropy of dephasing. Measurement of the GP for the set of initial states (12) allows for the formulation of conclusions directly related to the microscopic

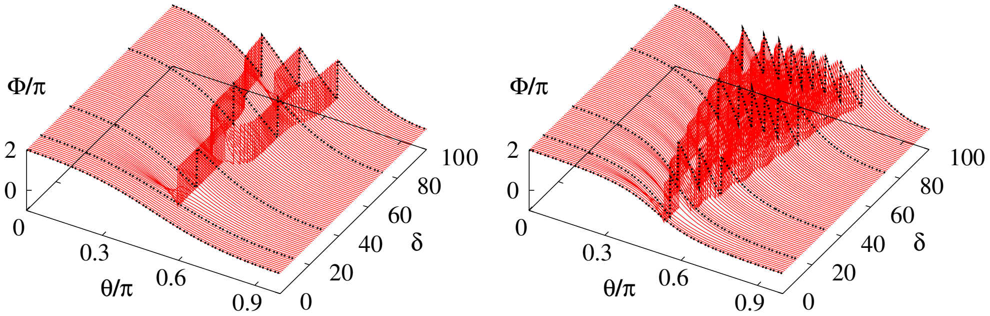

Figure 3. Bifurcations of the geometric phase $\Phi = \Phi(2\pi)$ for the asymmetric coupling parametrized by δ according to equation (24). The selected curves correspond to $\delta = 0, 10, 30, 70, 100$. Left panel: $\lambda = 0.01$. Right panel: $\lambda = 0.03$. In all cases $\mu_{\pm 1} = 0.1$.

properties, encoded in the Hamiltonian, of the open qubit system. The GP of a quantum system is a potential resource for quantum informatics by means of the holonomic quantum computing. The environment essentially present in each laboratory clearly, in general, spoils the phase. Robustness of the GP with respect to the environmental effects is a basic condition for an effective quantum computation. We have shown that even in the presence of the simplest environment, i.e. dephasing, the GP is very sensitive to its properties. We focused our attention on the possible asymmetry of the coupling of qubit levels to the bosonic environment. Within such a model, we have shown that the asymmetry of dephasing results in bifurcations and formation of jumps in the geometric phase.

6. Acknowedgment

The work is supported by the Polish Ministry of Science and Higher Education under the grant N 202 131 32/3786.

References

[1] Dajka J, Mierzejewski M and Łuczka J 2008 *J. Phys. A: Math. Theor.* **41** F012001
[2] van Hove L 1952 *Physica* **18** 145
[3] Friedrichs K O 1948 *Commun. Pure Appl. Math.* **1** 361
[4] Berry M V 1984 *Proc. R. Soc. Lond.* A **329** 45
Wilczek F and Zee A 1984 *Phys. Rev. Lett.* **52** 2111
[5] Aharonov Y and Ananadan J 1987 *Phys. Rev. Lett.* **58** 1593
[6] Chruściński D and Jamiołkowski A 2004 *Geometric Phases in Classical and Quantum Mechanics* (Boston: Birkhauser)
[7] Uhlmann A 1976 *Rep. Math. Phys.* **9** 273
[8] Sjöqvist E *et al* 2000 *Phys. Rev. Lett.* **85** 2845
[9] Zanardi P and Rasetti M 1999 *Phys. Lett.* A **264** 94
[10] Jones J A, Vedral V, Ekert A and Castagnoli 2000 *Nature* **403** 869
[11] Sarandy M S and Lidar D A 2005 *Phys. Rev. Lett.* **95** 250503
[12] Łuczka J 1990 *Physica* A **167** 919
Dajka J, Mierzejwski M and Łuczka J 2008 *Phys. Rev.* A **77**
[13] Spohn H 1987 *Commun. Math. Phys.* **123** 277
Alicki R 2004 *Open Syst. Inf. Dyn.* **11** 53
[14] Tang J 1994 *Chem. Phys.* **188** 143
[15] Markham J J 1956 *Rev. Mod. Phys.* **31** 956
Lin S H 1966 *J. Chem. Phys.* **44** 3759

[16] Gardiner S A, Cirac J I and Zoller P 1997 *Phys. Rev. Lett.* **97** 4790
[17] Montangero S, Romito A, Benenti G and Fazio R 2005 *Europhys. Lett.* **71** 893
[18] Pozzo E N and Dominguez D 2007 *Phys. Rev. Lett.* **98** 057006
[19] Reina J H, Quiroga L and Johnson N F 2002 *Phys. Rev.* A **65** 032326
Dajka J, Mierzejewski M and Łuczka J 2007 *J. Phys. A: Math. Theor.* **40** F879
[20] Berezin F A 1966 *Method of Second Quantization* (New York: Academic)
[21] Dajka J and Łuczka J 2008 *Phys. Rev.* A **77** 062303
[22] Perelomov A 1986 *Generalized Coherent States and Applications* (Berlin: Springer)
[23] Vourdas A 2006 *J. Phys. A: Math. Gen.* **39** R65
[24] Tong D M, Sjöqvist E, Kwek L C and Oh C H 2004 *Phys. Rev. Lett.* **93** 080405